Eigenvalues and Eigenvectors

If we have a $n \times 1$ column vector $\mathbf{v}$ and multiply it on the left by a $n \times n$ matrix A, then we will obtain another $n \times 1$ column vector $A\mathbf{v}$. In general this new vector will not be parallel to $\mathbf{v}$ but for certain vectors it may turn out that $\mathbf{v}$ and $A\mathbf{v}$ are parallel. That is, it may happen that

$$A\mathbf{v} = \lambda\mathbf{v} \text{ for some number } \lambda.$$

The vectors $\mathbf{v}$ for which this happens and the corresponding λ's are very special and we say that λ is an *eigenvalue* of A with $\mathbf{v}$ the corresponding *eigenvector*.

In order to find the eigenvalues λ of a matrix A we solve the *characteristic equation*

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix of the same size as A.
Note that an equivalent form of the characteristic equation is

$$\det(\lambda I - A) = 0,$$

and this will give exactly the same eigenvalues as $\det(A - \lambda I) = 0$, so it doesn't matter which one you use.
For an $n \times n$ matrix $\det(A - \lambda I)$ will be a polynomial of degree n, so let us first look at an example when $n = 2$.

Example: Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$.
We start by writing down the characteristic equation. In this case it is

$$\det\left(\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

We can write this as

$$\det\begin{pmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{pmatrix} = 0$$

and on calculating the determinant we obtain

$$(1-\lambda)(-4-\lambda) - 6 = 0$$

or

$$\lambda^2 + 3\lambda - 10 = 0.$$

Thus

$$(\lambda - 2)(\lambda + 5) = 0,$$

so that the eigenvalues are

$$\lambda = 2 \text{ and } \lambda = -5.$$

Once we have found the eigenvalues we have to find the eigenvectors corresponding to each one.

Example: Find the eigenvectors corresponding to the eigenvalues found above.

$\lambda = 2$: we first form the *eigenvector equation*

$$A\mathbf{v} = \lambda\mathbf{v},$$

where we let

$$\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $\lambda = 2$, this is

$$\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}.$$

On multiplying this out we obtain

$$\begin{pmatrix} x + 3y \\ 2x - 4y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

So we obtain the two equations

$$x + 3y = 2x \quad \text{and} \quad 2x - 4y = 2y.$$

Both these equations reduce to $x = 3y$, so that any non-zero vector of the form $\begin{pmatrix} 3a \\ a \end{pmatrix}$ will be an eigenvector corresponding to the eigenvalue $\lambda = 2$.
If we just want one eigenvector, then we can let $a = 1$, say, to obtain the eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$\lambda = -5$: In this case the eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ becomes

$$\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}.$$

On multiplying this out we obtain

$$\begin{pmatrix} x + 3y \\ 2x - 4y \end{pmatrix} = \begin{pmatrix} -5x \\ -5y \end{pmatrix},$$

which yields the two equations

$$x + 3y = -5x \quad \text{and} \quad 2x - 4y = -5y.$$

Both these equations reduce to $y = -2x$, so that any non-zero vector of the form $\begin{pmatrix} a \\ -2a \end{pmatrix}$ will be an eigenvector corresponding to the eigenvalue $\lambda = -5$.
If we just want one eigenvector, then we can let $a = 1$, say, to obtain the eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

In summary, the eigenvalues of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$ are 2 and -5 with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

As a check, note that the eigenvector equation holds in both cases:

$$\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -5\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$